**KEY TERMS & MAIN RESULTS – DISCRETE MATHEMATICS**

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| **Key terms** | **Examples** | **Exercises – Do yourself** | |
| Chapter 1 – Logic & Proofs | | | |
| Propositions | **Ex.** Determine whether the **proposition** is TRUE or FALSE.  a/ 1 + 1 = 2 and 2 + 2 = 1.  b/ 1 + 1 = 2 or 2 + 2 = 1  c / 1 + 1 = 2 if and only if 2 + 2 = 1.  d/ 1 + 1 = 2 if 2 + 2 = 1.  e/ If it is snowing, then it is snowing.  ***Solution.***  a/ FALSE (T ∧ F)  b/ TRUE (T ∨ F)  c/ FALSE (T↔F)  d/ TRUE (F →T)  e/ TRUE (p → p) | **1/** Determine whether the **proposition** is TRUE or FALSE.  a/ 1 + 1 = 2 if and only if pigs can fly.  b/ I am a superman if 1 + 1 = 2.  c/ If 1 + 1 = 2 or 1 + 1 = 3 but not both, then I can fly.  d/ For every nonnegative integer, n, the value of n2 + n + 41 is prime. | |
| Truth tables | **Ex.** Write the **truth table** for the proposition.  ***Solution.***   |  |  |  |  | | --- | --- | --- | --- | | p | *q* | *r* |  | | T | T | T | T | | T | T | F | T | | T | F | T | F | | T | F | F | T | | F | T | T | T | | F | T | F | F | | F | F | T | F | | F | F | F | F | | **2/** Construct the **truth tables** for the propositions:  a/(p ∧¬q) ∨ (¬p ∧ q)  b/  c/  d/ p → (q ⊕ p) | |
| Connectives / Operations | **Ex.**Let p and q be the propositions  p : It is below freezing.  q : It is snowing.  Write these propositions using p ***and*** q and logical **connectives** (including negations).  a/ It is below freezing ***but not*** snowing.  b/ It is ***either*** snowing ***or*** below freezing (or both).  c/ That it is below freezing is ***necessary and sufﬁcient*** for it to be snowing.  ***Solution.***  a/ p ∧¬q  b/ p ∨ q  c/ p ↔ q | **3/**Let p, q, and r be the propositions:  p :You get an A on the ﬁnal exam.  q :You do every exercise in this book.  r :You get an A in this class.  Write these propositions using p, q, and r and logical**connectives** (including negations).  a/ You get an A in this class, but you do not do every exercise in this book.  b/ You get an A on the ﬁnal, you do every exercise in this book, and you get an A in this class.  c/ To get an A in this class, it is necessary for you to get an A on the ﬁnal.  d/ Getting an A on the ﬁnal and doing every exercise in this book is sufﬁcient for getting an A in this class. | |
| Tautology | **Ex.**Determine whether this proposition is a **tautology**:  ***Solution.***  Truth table of :   |  |  |  |  |  | | --- | --- | --- | --- | --- | | p | q | p→q | (p→q)∧¬q | (p→q)∧¬q →¬p | | T | T | T | F | T | | T | F | F | F | T | | F | T | T | F | T | | F | F | T | T | T |  * is a **tautology**. | **4/**Determine whether each of these propositions is a **tautology**:  a/  b/ (p → q) ∨ (q → p)  c/  d/ ¬(p →¬p) → q. | |
| If-then  Necessary  Sufficient | **Ex.**Write each of these statements in the form “if p, then q” in English.  a/ To get a good grade it is **necessary** that you study.  b/ Studying is **sufficient** for passing.  ***Solution.***  a/ If you get a good grade, then you study. (Equivalently, if you don't study, then you don't get a good grade.)  b/ If you study, then you pass. | **5/**Write each of these statements in the form “if p, then q” in English.  a/ It is **necessary** to walk 8 miles to get to the top of  Long’s Peak.  b/ A **sufﬁcient** condition for the warranty to be good is that you bought the computer less than a year ago.  c/ I will remember to send you the address **only if** you send me an e-mail message.  (Hint: “**if** p, then q” can be written as “p **only if** q”). | |
| If and only if | **Ex.**Write each of these propositions in the form “p if and only if q” in English.  a/ If it is hot outside, you buy an ice cream cone, and if you buy an ice cream cone, it is hot outside.  b/ For you to win the contest it is **necessary and sufﬁcient** that you have the only winning ticket.  c/ If you watch television, your mind will decay, and **conversely**.  ***Solution.***  a/ It is hot outside **if and only if** you buy an ice cream cone.  b/ You win the contest if and only if you have the only winning ticket.  c/ Your mind will decay **if and only if** you watch television. | **6/**Write each of these propositions in the form “p if andonly if q” in English.  a/ If you read the newspaper every day, you will be in formed, and **conversely**.  b/ For you to get an A in this course, it is **necessary and sufﬁcient** that you learn how to solve discrete mathematics problems.  c/ It rains if it is a weekend day, and it is a weekend day if it rains. | |
| Negation | **Ex.**Find the **negation** of the propositions  a/ It is Thursday **and** it is cold.  b/ I will go to the play **or** read a book.  c/ If it is rainy, then we go to the movies.  ***Solution.***  a/ It is not Thursday **or** it is not cold.  (Keep in mind, )  b/ I won’t go to the play **and**I won’t read a book.  (Keep in mind, )  c/ It is not rainy but we don’t go to the movies.  (Keep in mind, ) | **7/** Find the **negation** of the propositions.  a/ If you study, then you pass.  b/ Alex and Bob are absent.  c/ He is young or strong. | |
| Equivalence | **Ex1.**a/ Write a proposition **equivalent** to that uses only *p*, *q,*¬and the connective ∧.  b/ Write a proposition **equivalent** to .  ***Solution.***  a/    So,.  b/    (Keep in mind, )  **Ex2.** Determine whether two propositions are **equivalent**.  a/  and  b/ and  ***Solution.***  a/ Use a truth table   |  |  |  |  | | --- | --- | --- | --- | | p | q | p→q | ¬p →¬q | | T | T | T | T | | T | F | F | T | | F | T | T | F | | F | F | T | T |  * NOT EQUIVALENT.   b/ Starting from the right-hand side,🡺 EQUIVALENT. | **8/**a/ Write a proposition **equivalent** to that uses only *p*, *q,*¬and the connective ∧.  b/ Write a proposition **equivalent** to .  c/ Write a proposition **equivalent** to (¬p ∨¬q) → (p ∧¬q).  **9/** Determine whether two propositions are **equivalent**.  a/  and  b/ and  c/ | |
| Predicates  Quantifiers | **Ex1.** What is the truth values of these propositions? (the domain for variable x is {-3, -2, -1, 0, 1, 2})  a/  b/  c/  ***Solution.***  a/ FALSE (counter example: x = -2)  b/ FALSE (counter example: x = 0)  c/ TRUE (no counter example)  **Ex2.**Suppose *P*(*x*,*y*) is a predicate and the universe for the variables x and y is {1,2,3}.  Suppose *P*(1,3), *P*(2,1), *P*(2,2), *P*(2,3), *P*(2,3), *P*(3,1), *P*(3,2) are true, and *P*(*x*,*y*) is false otherwise.  Determine whether the following statements are true.  a/  b/  c/  ***Solution.***  a/ TRUE (we can see P(1, 3), P(2, 2), P(3, 2) are true 🡺 for each x in {1, 2, 3}, there is at least one y in {1, 2, 3}.)  b/ FALSE (we can see that no y in {1, 2, 3} for all x in {1, 2, 3}, details are in below:   * y = 1: P(2, 1), P(3, 1) are true only (true with x = 2, 3, all x in {1, 2, 3}). * y = 2: P(2, 2), P(3, 2) are true only. * y = 3: P(1, 3), P(2, 3) are true only.   c/ TRUE   * x = 1: P(1, 3) 🡪 P(3, 1) * x = 2: P(2, 2) 🡪 P(2, 2) * x = 3: P(3, 1) 🡪 P(1, 3)   **Ex3.** Find the negation of each of these statements.  a/  b/  c/ | **10/**What is the truth values of these propositions? (the domain for variable x is the set of all real numbers.)  a/  b/  c/  **11/**Suppose *P*(*x*,*y*) is a predicate and the universe for the variables x and y is {1,2,3}.  Suppose *P*(1,3), *P*(2,1), *P*(2,2), *P*(2,3), *P*(2,3), *P*(3,1), *P*(3,2) are true, and *P*(*x*,*y*) is false otherwise.  Determine whether the following statements are true.  a/  b/  **12/**Find a ***negation*** of each of these statements:   1. ∀x(P(x) → Q(x)) 2. ∃x(P(x) ∧¬Q(x)) 3. ∀x∃y(¬P(x, y) ∨¬Q(x, y)   ∀x ∈ R (x < 2 → x2< 4) | |
| Translation | **Ex.**Suppose the variable *x* represents students and *y* represents courses, and:   * *A*(*y*): *y* is an advanced course * *M*(*y*): *y* is a math course * *F*(*x*): *x* is a freshman * *B*(*x*): *x* is a full-time student * *T*(*x, y*): student *x* is taking course *y*.   Write these statements using these predicates and any needed quantifiers.  a/ Linh is taking MAD101.  b/ No math course is an advanced course.  c/ Every freshman is a full-time student.  d/ There is at least one course that every full-time student is taking.  ***Solution.***  a/ T(Linh, MAD101)  b/  or equivalently,  c/  d/ . | **13/**Suppose the variable *x* represents students and *y* represents courses, and:   * *A*(*y*): *y* is an advanced course * *M*(*y*): *y* is a math course * *F*(*x*): *x* is a freshman * *B*(*x*): *x* is a full-time student * *T*(*x, y*): student *x* is taking course *y*.   Write these statements using these predicates and any needed quantifiers.  a/ Nam is taking a math course.  b/ There are some freshmen who are not taking any course.  c/ There are some full-time students who are not taking any advanced course. | |
| Arguments  Valid/invalid  Rules of inference | **Ex.**Determine whether the following argument is valid.  “Rainy days make gardens grow. Gardens don't grow if it is not hot. It always rains on a day that is not hot.  Therefore, if it is not hot, then it is hot.”  ***Solution.***  Consider the statements:  r : it a rainy day  g: gardens grow  h: it is hot  Then,   * Rainy days make gardens grow can be written as “r → g” (1) * “Gardens don't grow if it is not hot” is denoted by “h →¬g” (2) * “It always rains on a day that is not hot” becomes “¬h → r” (3)   From (3), ¬h → r and from (1), r→ g. So, ¬h → g (4) can be drawn.  From (2), h →¬g, this is equivalent to g →¬h (5).  From (4) and (5), ¬h → g and g →¬h, we can conclude that ¬h → h, or in words “if it is not hot, then it is hot”.   * VALID ARGUMENT. | **14/**Determine whether the following argument is valid.  Dong is anAI Major or a CS Major but not both.If he does not know discrete math, he is not anAI Major.If he knows discrete math, he is smart.He is not a CS Major. Therefore, he is smart. | |
| ***Applications.***   1. **Logic Circuits. (readings – pages \_\_\_\_)**   Find the output of each of these combinatorial circuits.     1. The goal of this exercise is to ***translate*** some assertions about binary strings into logic notation.  * The domain of discourse is the set of all ﬁnite-length binary strings: λ, 0, 1, 00, 01, 10, 11, 000, 001, . . . . (Here λ denotes the *empty string*.) * Consider a string like **10x1y**, if the value of x is 110 and the value of y is 11, then the value of **10x1y** is the binary string 10110111. * Here are some examples of formulas and their English translations. Names for these predicates are listed in the third column so that you can reuse them in your solutions (as I do in the deﬁnition of the predicate NO-1S below).      1. x consists of three copies of some string. 2. x is an even-length string of 0’s.   x does not contain both a 0 and a 1. | | | |
| Chapter 2 – sets, sequences, sums | | | |
| Sets  Elements  Empty set  Subsets | **Ex1.** Determine whether each of these statements is true or false.  a/ 2 ∈{2,{2}}  b/ 2 ∈{{2},{{2}}}  c/ ∅∈{0}  d/ ∅∈{∅, {∅}}  e/ ∅⊆ {0}  ***Solution.***  a/ True  b/ False  c/ False  d/ True  e/ True | **15/**Determine whether each of these statements is true or false.  a/ 2 ∈{{{2}}}  b/ 2 ∈{{2},{2,{2}}}  c/ ∅∈{x}  d/ ∅⊆{x} | |
| Cardinality of a set | **Ex.** What is the **cardinality** of each of these sets?  a/ {a, {a}}  b/ {∅, a, {a, {a}}}  ***Solution.***  a/ |{a, {a}}| = 2  b/ |{∅, a, {a, {a}}}| = 3 | **16/**What is the **cardinality** of each of these sets?  a/ {∅, {∅}}  b/ { {a, {a}, b} } | |
| Power set | *The* ***power set*** *of a set A, denoted by P(A), is the set of all subsets of A.*  For example, if A = {1, 2}, then the power set of A is the set P(A) = {∅, {1}, {2}, {1, 2}}.  If A contains n elements, P(A) contains 2n elements.  **Ex1.** Determine whether each of these sets is the **power set** of a set, where a and b are distinct elements.  a/ {∅, {a}}  b/ {∅, {a}, {∅,a}}  c/ {∅, {a}, {b}, {a, b}}  ***Solution.***  a/ {∅, {a}} is the power set of the set {a}.  b/ {∅, {a}, {∅,a}} cannot be a power set of any set.  c/ {∅, {a}, {b}, {a, b}} is the power set of the set {a, b}.  **Ex2.**How many elements does each of these sets have?  a/ P({a, {a}})  b/ P({∅, a, {a}, {{a}}})  c/ P(P(∅))  ***Solution.***  a/ |{a, {a}}| = 2 🡺 | P({a, {a}})| = 22 = 4.  b/ |{∅, a, {a}, {{a}}}| = 4  🡺 |P({∅, a, {a}, {{a}}})| = 24 = 16.  c/ |∅| = 0 🡺 |P(∅)| = 20 = 1 🡺 | P(P(∅))| = 21 = 2 | **17/**Determine whether each of these sets is the **power set** of a set?  a/ ∅  b/ {∅}  c/ {∅, {a}, {∅}}  d/ {∅, {{1}}, {2}, {{1}, 2}}  **18/**How many elements does each of these sets have?  a/ P({∅, {a}})  b/ P({a, {a}, {a, {a}}})  c/ P(P({∅})) | |
| Union ∪  Intersection ∩  Difference –  Symmetric difference ⊕  Complement | **Ex1.**Prove that, for all sets A, B:  a/  b/ A – B ⊆ A.  c/ A = (A – B) ∪ (A ∩ B)  ***Solution.***  a/ We use a membership table:   |  |  |  |  |  | | --- | --- | --- | --- | --- | | A | B |  | A – B |  | | 1 | 1 | 0 | 0 | 0 | | 1 | 0 | 1 | 1 | 1 | | 0 | 1 | 0 | 0 | 0 | | 0 | 0 | 1 | 0 | 0 |   Based on the agreement of two latest columns, an element belongs to A – B if and only if it belongs to .  So,.  b/ Membership table:   |  |  |  |  | | --- | --- | --- | --- | | A | B | A – B | A | | 1 | 1 | 0 | 1 | | 1 | 0 | 1 | 1 | | 0 | 1 | 0 | 0 | | 0 | 0 | 0 | 0 |   From the table, if an element belongs to A – B (the corresponding number is 1), then it also belongs to A (the corresponding number is also 1).  **Ex2.**Find the sets A and B if A − B ={1, 5, 7, 8}, B − A = {2, 10}, and A ∩ B ={3, 6, 9}.  ***Solution.***  From Ex1 (c):   * A = (A – B) ∪ (A∩B) = {1, 5, 7, 8, 3, 6, 9} * B = (B – A) ∪(A∩B) = {2, 10, 3, 6, 9}. | **19/**Show that if A and B are sets with A ⊆ B, then  a/ A ∪ B = B  b/ A ∩ B = A  c/ A ∩ B ⊆ A  d/ A ⊕ B = B – A  e/ .  **20/**Find the sets A and B if A ⊆ B and A ∪ B = {1, 3, 4, 5, 7, 9}, and A ∩ B = {3, 4, 7}.  **21/**Find the sets A and B if A − B ={2, 3, 5, 7}, B − A = {1, 4}, and A ∩ B ={8, 6}. | |
| AB | **Ex.**Given the sets C = {red; blue; yellow} and S = {small, medium, large}.  a/ Construct Cartesian product CS.  b/ What is the **cardinality** of the set CS ? How many **subsets** does CS have?  ***Solution.***  a/ CS = {(red, small), (red, medium), (red, large), (blue, small), (blue, medium), (blue, large), (yellow, small), (yellow, medium), (yellow, large)}.  b/ |CS | = 3⋅3 = 9🡺 AB has 29 subsets. | **22/**Given the sets A = {0, 1}.  a/ Construct the set AA.  b/ Find the **complement** of the set {(0, 1)} in AA.  c/ What is the **cardinality** of the set AA? List all **subsets** of AA. | |
| Set representation | **Ex1.** Let U = {a, b, c, d, e, f, g} be the universal set. Find the bit string representing the subset A = {a, c, d, g}.  ***Solution.***   |  |  |  |  |  |  |  |  | | --- | --- | --- | --- | --- | --- | --- | --- | | U | a | b | c | d | e | f | g | | U | 1 | 1 | 1 | 1 | 1 | 1 | 1 | | A | 1 | 0 | 1 | 1 | 0 | 0 | 1 |   **Ex2.**Let U = {1, 2, 3, 4, 5, 6, 7, 8}.  Given the subsets A = {1, 2, 3, 5, 7}, B = {2, 4, 5}. Find the bit string representing the subset A – B.  ***Solution.***   |  |  |  |  |  |  |  |  |  | | --- | --- | --- | --- | --- | --- | --- | --- | --- | | U | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | | A | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | | B | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | | A-B | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | | **23/**Suppose that the universal set is U ={1, 2, 3, 4, 5, 6, 7, 8}. Express each of these sets with bitstrings.  a/ {3, 4, 5}  b/ {1, 3, 6, 8}  c/ {1, 2, 3, 5} ⊕ {2, 3, 4, 6, 7}  **24/** Let U = {a, b, c, d, e, f, g} be the universal set. Suppose A and B are sets given by bit strings 1010101 and 1100111. List all elements in the set . | |
| functions | **Ex1.** Determine which rules are functions.  a/ f: Z → Z; f(x) = 1/(2x-1)  b/ f: Z → R; f(x) = 1/(2x-1)  c/ f: R → R; f(x) = 1/(2x-1)  ***Solution.***  a/ f: Z →Z; f(x) = 1/(2x-1)  This rule is not a function, because f(2) = 1/3 does not belong to the set Z (the set of integers).  b/ f: Z → R; f(x) = 1/(2x-1)  This rule is a function, we determine exactly one output value for every input value.  c/ f: R → R; f(x) = 1/(2x-1)  This rule is not a function because f(1/2) is not defined.  **Ex2.**Determine whether f is a function from the set of all bit strings to the set of integers if **f(S) is the position of a 0 bit** in S.  ***Solution.***  Consider the string S = “10011” as an input,  f(S) = the position of a 0 bit in S 🡺 f(10011) can be 2 or 3 🡺 f is NOT a function. | **25/** Determine which rules are functions.  a/ f: Z →Z; f(x) = 1/(x2 - 2)  b/ f: Z → R; f(x) = 1/(x2-2)  c/ f: R →R; f(x) = 1/(x2-2)  **26/**Determine whether f is a function from the set of all bit strings to the set of integers if f(S) is the number of 0 bits in S.  **27/**Let R be the set {(a, b) | a - 1 = b or b - 1 = a}, where a and b are in {-2, -1, 0, 1, 2}.  a/ List all ordered pairs of R.  b/ Is R a function? Explain your answer. | |
| One-to-one  Onto  Bijection  Inverse functions (f-1)  Invertible | **Ex1.**a/ Determine whether the function from N = {0, 1, 2, …} to N is **one-to-one.**  a/ f(n) = (n – 1)2  b/ Determine whether the function from Z = {…, -2, -1, 0, 1, 2, …} to N = {0, 1, 2, …} is **one-to-one.**    ***Solution.***  a/ f(2) = f(0) = 1 🡺 f is not one-to-one.  b/   * If n, m are different negative integers 🡺f(n) ≠ f(m) because f(n) = -2n ≠ -2m = f(m). * If n, m are different non-negative integers 🡺f(n) ≠ f(m) because 2n+1 ≠ 2m + 1. * If n is negative and m is non-negative 🡺 f(n) = -2n (even) and f(m) = 2m + 1 (odd) 🡺 f(n) ≠ f(m)   🡺  🡺f is one-to-one.  **Ex2.**Determine whether the function f from the set of all bit strings to the set of integers is one-to-one if f(S) is the number of 1bits in S.  ***Solution.***  f(01011) = f(1110) = 3 🡺 f is not one-to-one.  **Ex3.**a/ Determine whether the function f(n) = (n – 1)2 from N = {0, 1, 2, …} to N is **onto.**  b/ Determine whether the function from Z = {…, -2, -1, 0, 1, 2, …} to N = {0, 1, 2, …} is **onto.**    ***Solution.***  a/ Because f(n) = (n-1)2≠ 2 for all values of n 🡺 f is not onto.  b/ Because f(n) ≠ 0 for all n 🡺 f is not onto.  **Ex4.**Determine whether each of these functions is a **bijection** from R to R. In case f is a bijection, find the inverse function f-1.  a/ f(x) = −3x + 4  b/ f(x) = −3x2 + 7  ***Solution.***  a/ For every y in R, we can find **exactly one** x in R such that y = −3x + 4. In this case, x = (y – 4)/(-3).  And the **inverse function** is f-1(y) = (y – 4)/(-3).  b/ For some y in R, we cannot find x (or can find more than one values of x) in R such that y = -3x + 4. For example, no value of x in R such that 10 = -3x2 + 7 or 1 = -x2.   * f is not a **bijection**. | **28**a/ Determine whether the function from f(n) = (n + 1)2N = {0, 1, 2, …} to N is **one-to-one.**  b/ Determine whether the function from Z = {…, -2, -1, 0, 1, 2, …} to N = {0, 1, 2, …} is **one-to-one.**  **29/**Determine whether the function f from the set of all bit strings to the set of integers is one-to-one if f(S) is the number of 0 bits in S.  **30**a/ Determine whether the function from f(n) = (n + 1)2 N = {0, 1, 2, …} to N is **onto.**  b/ Determine whether the function from N = {0, 1, 2, …} to Z = {…, -2, -1, 0, 1, 2, …} is **onto**    **31a/** List all ***functions*** from {, } to {SHOOT, PASS, SPRINT}.  b/ List all ***one-to-one*** functions from {, } to {SHOOT, PASS, SPRINT}.  c/ List all ***onto*** functions from {, } to {SHOOT, PASS, SPRINT}.  **32/** Determine whether each of these functions is a **bijection** from R to R. In case f is a **bijection**, find the **inverse function** f-1.  a/ f(x) = 2x − 5  b/ f(x) = (x – 3)(x + 1) | |
| Composite function | **Ex1.** Find fg and gf, where f(x) = x2 + 1 and g(x) =x + 2, are functions from R to R.  ***Solution.***   * (fg)(x) = f(g(x)) = f(x+2) = (x+2)2 + 1 * (gf)(x) = g(f(x)) = g(x2 + 1) = (x2 + 1) + 2 = x2 + 3.   **Ex2.**Let f = {(a, 1); (b, 3); (c, 2)} be a function from {a, b, c} to {1, 2, 3}.  a/ Find f-1.  b/ Find ff-1 and f-1f.  ***Solution.***  a/ f-1 = {(1, a); (3, b); (2, c)}  b/ ff-1 = {(1, 1); (2, 2); (3, 3)}  and f-1f = {(a, a); (b, b); (c, c)}. | **33/**Find fg and gf, where f(x) = 2x + 1 and g(x) = 1 – x3, are functions from R to R.  **34/** Let g = {(1, c); (2, b); (3, a)} be a function from {1, 2, 3} to {a, b, c}.  a/ Find g-1.  b/ Find gg-1 and g-1g. | |
| Sequences | **Ex1.** List the ﬁrst 6 terms of each of these sequences.  a/ the sequence that lists each positive integer three  times, in increasing order  b/ the sequence whose nth term is 2n – n2  c/ the sequence whose ﬁrst term is 2, second term is 4, and each succeeding term is the sum of the two previous terms.  ***Solution.***  a/ 111, 222, 333, 444, 555, 666  b/ 1, 0, -1, 0, 7, 28  c/ 2, 4, 6, 10, 16, 26  **Ex2.**Find the ﬁrst four terms of the sequence deﬁned by each of these recurrence relations and initial conditions.  a/ an = −2an−1, a0 = −1  b/ an = an−1 − an−2, a0 = 2, a1 = −1  c/ an = an−1, a0 = 5.  ***Solution.***  a/ a0 = -1  a1 = -2a0 = -2.(-1) = 2  a2 = -2a1 = -2(2) = -4  a3 = -2a2 = -2(-4) = 8  b/ a0 = 2, a1 = -1  a2 = a1 – a0 = -1 – 2 = -3  a3 = a2 – a1 = -3 – (-1) = -2  c/ a0 = 5  a1 = a0 = 5  a2 = a1 = 5  a3 = a2 = 5 | **35/**List the ﬁrst 6 terms of each of these sequences.  a/ the sequence whose nth term is the sum of the ﬁrst n  odd positive integers  b/ the sequence whose nth term is n! − 2n  c/ the sequence whose ﬁrst two terms are 1 and 5 and each succeeding term is the sum of the two previous terms.  **36/** Find the ﬁrst four terms of the sequence deﬁned by each of these recurrence relations and initial conditions.  a/ an = −an−1, a0 = 5  b/ an = an−1 − n, a0 = 4  c/ an = an-2, a0 = 3, a1 = 5 | |
| Special sums | Special sum:    **Ex1.** Find the value of each of these sums.  a/  b/  c/  d/  ***Solution.***  a/  b/  c/  d/    **Ex2.** Compute each of these double sums.  a/  b/  c/  ***Solution.***  a/    b/    c/ | **37/** Find the value of each of these (double) sums.  a/  b/  c/  d/  e/ | |
| Chapter 3 – Algorithms & Integers | | | |
| Algorithms | **Ex1.** List all the steps used to search for 9 in the sequence 2, 3, 4, 5, 6, 8, 9, 11 using a **linear search**. How many comparisons required to search for 9 in the sequence?  ***Solution.***  Below is the linear search algorithm in pseudocode  procedure linear search(x: integer, a1, a2,..., an: distinct integers)  i := 1  while (i ≤ n and x = ai )  i := i + 1  if i ≤ n then location := i  else location := 0  return location{location is the subscript of the term that equals x, or is 0 if x is not found}  All the steps used to search for 9 using a linear search:  i = 1  (1 ≤ 8 and 9 ≠ 2) 🡺 i:=i+1 = 2  i = 2  (2 ≤ 8 and 9 ≠ 3) 🡺 i:= i+1 = 3  i = 3  (3 ≤ 8 and 9 ≠ 4) 🡺 i:= i+1 = 4  i = 4  (4 ≤ 8 and 9 ≠ 5) 🡺 i:= i+1 = 5  i = 5  (5 ≤ 8 and 9 ≠ 6) 🡺 i:= i+1 = 6  i = 6  (6 ≤ 8 and 9 ≠ 8) 🡺 i:= i+1 = 7  i = 7  (7 ≤ 8 and 9 ≠ 9) // the condition is false  7 ≤ 9 🡺 location = 7.  Based on the steps above, there are 15 comparisons (≤, ≠) required. | **38/** List all the steps used to search for 8 in the sequence 3, 5, 6, 8, 9, 11, 13, 14 using a **binary search**. How many comparisons required to search for 8 in the sequence?  **39/**Josephus problem. This problem is based on an account by the historian Flavius Josephus, who was part of a band of 41 Jewishrebels trapped in a cave by the Romans during the Jewish Roman war of the ﬁrst century. The rebels preferred suicideto capture; they decided to form a circle and to repeatedlycount off around the circle, killing every third rebel left alive.  However, Josephus and another rebel did not want to be killedthis way; they determined the positions where they shouldstand to be the last two rebels remaining alive.  Devise an algorithm to determine the alive positions if the number of rebels is n and an alive rebel will be killed after counting to k (k < n). | |
| Big-O  Big-Omega  Big-theta | **Ex1.** In the table below, check ✓if the fact is true and check 🗶 otherwise.   |  |  |  |  | | --- | --- | --- | --- | | function | = O(x2) | = Ω(x2) | = Θ(x2) | | 2x + 11 |  |  |  | | x2 + 3x + 1 |  |  |  | | x2logx + 2018 |  |  |  | | x3 – 5x2 +3 |  |  |  |   ***Solution.***   |  |  |  |  | | --- | --- | --- | --- | | function | = O(x2) | = Ω(x2) | = Θ(x2) | | 2x + 11 | ✓ | 🗶 | 🗶 | | x2 + 3x + 1 | ✓ | ✓ | ✓ | | x2logx + 2018 | 🗶 | ✓ | 🗶 | | x3 – 5x2 +3 | 🗶 | ✓ | 🗶 |   **Ex2.**Find the least integer k such that is O(xk).  ***Solution.***      So,  In other hand, x2logx is O(x3) 🡺 the least integer k is 3.  **Ex3.** a/ Show that log10n is O(logn)  b/ Show that log(n!) is O(nlogn).  ***Solution.***  a/ log10n = log102.logn 🡺 log10n is O(logn)  b/ log(n!) = log(1⋅2⋅3⋅⋅⋅n) ≤ log(n⋅n⋅n⋅⋅⋅n) = log(nn) = nlogn   * log(n!) is O(nlogn). | **40/**Determine whether each of these functions is O(x2).  a/ f(x) = 3x + 7  b/ f(x) = log(x3) + 2x  c/ f(x) = (2x3 + x2 log x)/(x+2)  d/ f(x) = 2x + 1  **41/** Find the least integer k such that f(x) is O(xk) for each of these functions.  a/ f(x) = 2x2 + x2log x  b/ f(x) = x3 + (logx)4  c/ f(x) = (xlogx + 3x)(x2 + 100x + 1).  **42/**Show that 1 + 2 + 3 + … + n is O(n2). | |
| Complexity of an algorithm | **Ex1.** Consider the **algorithm:**  procedure giaithuat(a1, a2, …, an : integers)  count:= 0  for i:= 1 to n do  if ai> 0 then count: = count + 1  print(count)  Give the **best big-O complexity** for the algorithm above.  ***Solution.***  With one “for loop” in the algorithm, the complexity of the algorithm is O(n).  **Ex2.** How much time does an algorithm take to solve a problem of size n if this algorithm uses 2n2 + 2n operations, each requiring 10−9seconds, with these values of n?  a/ 10  b/ 50  ***Solution.***  a/ n = 10 🡺 the algorithm uses 2.102 + 210 operations, each requiring 10−9 seconds  🡺need (2.102 + 210).10-9 = 0.000001224 seconds.  b/ n = 50 🡺 the algorithm uses 2.502 + 250 operations, each requiring 10−9 seconds 🡺 need (2.502 + 250). 10-9 = 1125900 seconds. | **43/**Consider the **algorithm:** procedure thuattoan(a1, a2, ..., an: positive real numbers).  m := 0  for i := 1 to n-1  for j := i + 1 to n  m := max(ai.aj, m)  Give the **best big-O complexity** for the algorithm above.  **44/** How much time does an algorithm take to solve a problem of size n if this algorithm uses 2n2 + 2n operations, each requiring 10−9 seconds, with these values of n?  a/ 30  b/ 100 | |
| Divide  Divisor  Division  Quotient  Remainder  mod and div | **Ex1.**Show that if a | b and b | c, then a | c, where a, b, c are integers.  ***Solution.***  a | b 🡺∃k∈Z (b = ka)  b | c 🡺∃m∈Z (c = mb)   * c = m(ka) = (mk)a, where mk is an integer * a | c.   **Ex2.** Prove or disprove that if ab | c, where a, b, and c are positive integers, then a | c and b | c.  ***Solution.***  ab | c 🡺∃k∈Z (c = kab)   * c = (kb)a and c = (ka)b, where ka, kb are integers * a | c and b | c.   **Ex3.** What are the quotient and remainder when  a/ 1001 is divided by 13?  b/ −111 is divided by 11?  ***Solution.***  a/ 1001 = 13.77 + 0   * quotient = 77 and remainder = 0.   b/ -111 = 11.(-11) + 10   * quotient = -11 and remainder = 10.   **Ex4.** Suppose a ***mod*** 4 = 3 and b ***mod*** 8 = 7, find ab ***mod*** 4.  ***Solution.***   * We have, b mod 8 = 7 🡺 b = 8k + 7, where k is an integer * b = 4(2k + 1) + 3 * b mod 4 = 3 * So, ab mod 4 = ((a mod 4).(b mod 4)) mod 4   = (3.3) mod 4 = 1. | **45/**Show that if a | b and b | a, then a = b or a = −b, where a, b are integers.  **46/**Prove or disprove that if a | bc, then a | b or a | c, where a, b, and c are positive integers and a ≠ 0.  **47/** What are the **quotient** and **remainder** when  a/ −1 is divided by 3?  b/ 3 is divided by 13?  c/ −123 is divided by 19?  **48/**Evaluate these quantities.  a/ −17 **mod** 2  b/ 144 **mod** 7  c/ −101 **div** 13  d/ 199 **div** 19  **49/**Suppose a ***mod*** 3 = 2 and b ***mod*** 6 = 4, find ab ***mod*** 3. | |
| Congruence | **Ex.** Decide whether each of these integers is **congruent to 5 modulo 17**.  a/ 80  b/ 103  c/ -29  d/ -122  ***Solution.***  Recall that *a is congruent to b modulo m* if and only if m **divides**a – b.  Or equivalently,  a/ 17 (80 – 5) 🡺 80 is not congruent to 5 modulo 17.  b/ 17 (103 – 5) 🡺 103 is not congruent to 5 modulo 17.  c/ 17 | (-29 – 5) 🡺 -29 is congruent to 5 modulo 17.  d/ 17 (-122 – 5) 🡺 -122 is not congruent to 5 modulo 17. | **50/**Decide whether each of these integers is **congruent to 3 modulo 7**.  a/ 37  b/ 66  c/ −17  d/ −67  **51/** Find an integer x in {0, 1, 2, …, 6} such that:  a/ 5.x ≡ 1 (mod 7).  b/ x.x2≡ 1 (mod 7). | |
| Encryption  Decryption  Hashing functions  Pseudo random numbers | **Ex1.** Suppose ***pseudo-random numbers*** are produced by using:  xn+1 = (3xn + 11) mod 13.  If x3 = 5, find x2 and x4.  ***Solution.***   * x4 = (3x3 + 11) mod 13   = (3.5 + 11) mod 13 = 0   * x3 = (3x2 + 11) mod 13   So, 5 = (3x2 + 11) mod 13  ⇔ 13 | (3x2 + 11 – 5)  ⇔ 13 | (3x2 + 6) (\*)  Note that x2 is in 0..12🡺 x2 = 11 is the solution of (\*).  **Ex2.** Using the function  f(x) = (x + 10) mod 26  to **encrypt** messages. Answer each of these questions.  a/ ***Encrypt*** the message STOP  b/ ***Decrypt*** the message LEI.  ***Solution.***   |  |  |  |  |  | | --- | --- | --- | --- | --- | | A | B | C | … | Z | | 0 | 1 | 2 |  | 25 |  |  |  |  |  | | --- | --- | --- | --- | | S | T | O | P | | 18 | 19 | 14 | 15 |  |  |  |  |  |  | | --- | --- | --- | --- | --- | | x | 18 | 19 | 14 | 15 | | f(x) = (x+10) mod 26 | 2 | 3 | 24 | 25 |  |  |  |  |  | | --- | --- | --- | --- | | 2 | 3 | 24 | 25 | | C | D | Y | Z |  * STOP has been encrypted to CDYZ.   b/ We will **decrypt** the message LEI using the inverse function f-1(x) = (x – 10) mod 26.   |  |  |  |  | | --- | --- | --- | --- | | Encrypted form | L | E | I | | x | 11 | 4 | 8 | | f-1(x) = (x – 10) mod 26 | 1 | 20 | 24 | | Original message | B | U | Y |   **Ex3.** Which memory locations are assigned by the **hashing function**h(k) = k mod 101 to the records of insurance company customers with these Social Security Numbers?  a/ 104578690  b/ 432222187  ***Solution.***  a/ h(104578690) = 104578690 mod 101 = 58.   * The memory location 58 is assigned to the customer with the Social Security number 104578690.   b/ h(501338753) = 501338753 mod 101 = 3.  So, the memory location 3 is assigned to the customer with the Social Security number 501338753. | **52/**Suppose ***pseudo-random numbers*** are produced by using:  xn+1 = (2xn + 7) mod 9.  a/ If x0 = 1, find x2 andx3.  b/ If x3 = 3, find x2 and x4.  **53**a/ **Encrypt** the message SELL using the function f(x) = (x + 21) mod 26  b/ **Decrypt** these messages “CFMV L” that were encrypted using the f(x) = (x + 17) mod 26.  **54/** A parking lot has 31 visitor spaces, numbered from 0 to30. Visitors are assigned parking spaces using the **hashing function** h(k) = k mod 31, where k is the number formed from the ﬁrst three digits on a visitor’s license plate. Which spaces are assigned by the **hashing function** to cars that have these ﬁrst three digits on their license plates: 317, 918, 007, 111? | |
| Prime, relatively prime  Gcd, lcm | **Ex1.** Which positive integers less than 30 are **relatively prime** to 30?  ***Solution.***  Recall that two positive integers a and b are called **relatively prime** if and only if their greatest common divisor is 1.  So, positive integers less than 30 are **relatively prime** to 30 are: 1, 7, 11, 13, 17, 19, 23, 29.  **Ex2.**The value of the **Euler φ-function** at the positive integer n, φ(n), is deﬁned to be the number of positive integers less than or equal to n that are **relatively prime**to n.  Find these values of the Euler φ-function.  a/ φ(6)  b/ φ(7)  ***Solution.***  a/ n = 6: positive integers less than or equal to 6 that are relatively prime to 6 are: 1, 5   * φ(6) = 2   b/ n = 7: positive integers less than or equal to 6 that are relatively prime to 6 are: 1, 2, 3, 4, 5, 6   * φ(7) = 6   **Ex3.** If the product of two integers is 273852711 and their **greatest common divisor** is 23345, what is their **least common multiple**?  ***Solution.***  If a and b are positive integers, then  **ab = gcd(a, b).lcm(a, b)**.  So, 273852711 = gcd(a, b).lcm(a, b) = 23345.lcm(a, b) 🡺 lcm(a, b) = 273852711/23345 = 24345(711) | **55/**Which positive integers less than 18 are **relatively prime** to 18?  **56/** Find these values of the **Euler φ-function**.  a/ φ(4)  b/ φ(5)  c/ φ(11)  **57/** If the product of two integers is 3072 and their **least common multiple** is 384, what is their **greatest common divisor**? | |
| Euclidean algorithm | **Ex.**Use the **Euclidean algorithm** to ﬁnd  a/ gcd(8, 28)  b/ gcd(100, 101).  ***Solution.***  a/ 28 mod 8 = 4 🡺gcd(8, 28) = gcd(4, 8)  8 mod 4 = 0 🡺gcd(4, 8) = gcd(0, 4) = 4.  b/ 101 mod 100 = 1 🡺gcd(100, 101) = gcd(1, 100)  100 mod 1 = 0 🡺gcd(1, 100) = gcd(0, 1) = 1. | **58/** Use the **Euclidean algorithm** to ﬁnd  a/ gcd(12, 18)  b/ gcd(111, 201). | |
| Integer representation  Decimal  Binary  Octal  Hexadecimal  Expansions  Base b expansions | **Ex1.** Convert 96 to  a/ a binary expansion.  b/ a base 5 expansion.  c/ a base 13 expansion.  ***Solution.***  a/ 96 = (1100000)2  b/   * 96 = 19.5 + 1 * 19 = 3.5 + 4 * 3 = 0.5 + 3 * 96 = 19.5 + 1 = (3.5 + 4).5 + 1 * 96 = 3.52 + 4.51 + 1.50 * 96 = (341)5   c/   * 96 = 7.13 + 5 * 7 = 0.13 + 7 * 96 = 7.131 + 5.130 * 96 = (75)13   **Ex2.**Convert each of the following expansions to **decimal expansion**.  a/ (1021)3  b/ (325)7  c/ (A3)12  ***Solution.***  a/ (1021)3 = 1.33 + 0.32 + 2.31 + 1.30 = 34  b/ (325)7 = 3.72 + 2.71 + 5.70 = 166  c/ (A3)12 = A.121 + 3.120 = 10.12 + 3 = 123. | **59/**Convert 69 to  a/ a binary expansion.  b/ a base 6 expansion.  c/ a base 9 expansion.  **60/** Convert each of the following expansions to **decimal expansion**.  a/ (401)5  b/ (12B7)13 | |
| **Applications:Check digits.**  1/ **UPCs.** Retail products are identiﬁed by their Universal Product Codes (UPCs). The most common form of a UPC has 12 decimal digits: the ﬁrst digit identiﬁes the product category, the next ﬁve digits identify the manufacturer, the following ﬁve identify the particular product, and the last digit is a **check digit**. The check digit is determined by the congruence  3x1 + x2 + 3x3 + x4 + 3x5 + x6 + 3x7 + x8 + 3x9 + x10 + 3x11 + x12 ≡ 0 (mod 10).  For example, if the ﬁrst 11 digits of a UPC are 79357343104, then the check digit is x12 = 2.  In fact, let x12 be check digit, we have  3 · 7 + 9 + 3 · 3 + 5 + 3 · 7 + 3 + 3 · 4 + 3 + 3 · 1 + 0 + 3 · 4 + x12 ≡ 0 (mod 10)  Simplifying, we have 98 + x12 ≡ 0 (mod 10) 🡺 x12 = 2.  a/ Find the check digit for the **USPS** money orders that have identiﬁcation number that start with these ten digits 7555618873 and 6966133421.  b/ Determine whether 74051489623 and 88382013445 are valid **USPS** money order identiﬁcation number.  2/ **Parity Check Bits**. Digital information is represented by bit string, split into blocks of a speciﬁed size. Before each block is stored or transmitted, an extra bit, called a **parity check** bit, can be appended to each block. The parity check bit xn+1 for the bit string x1x2...xn is deﬁned by xn+1 = x1 + x2 +···+ xn mod 2.  (It follows that xn+1 is 0 if there are an even number of 1 bits in the block of n bits and it is 1 if there are an odd number of 1 bits in the block of n bits). When we examine a string that includes a parity check bit, we know that there is an error in it if the parity check bit is wrong. However, when the parity check bit is correct, there still may be an error. For example, if we receive in a transmission the bit string 11010110, we ﬁnd that 1 + 1 + 0 + 1 + 0 + 1 + 1 ≡ 1 (mod 2), so the **parity check** is incorrect. So, we reject the string.  Suppose you received these bit strings over a communications link, where the last bit is a **parity check** bit. In which string are you sure there is an error?  a/ 00100111111  b/ 10101010101 | | | |
| Chapter 4 – Induction & Recursion | | | |
| Mathematical induction  Strong induction | **Ex1.**Prove the statement "6 divides n3 - n for all integers n ≥ 0", using ***mathematical induction*** method.  ***Solution.***  *Basis step*. The statement is true for n = 0, since 6 divides 0.  *Inductive step*.   * Suppose for every integer k ≥ 0, the statement is true, that is, "6 divides k3 - k" * We have, (k+1)3 - (k+1) = (k3 + 3k2 + 3k + 1) - (k + 1) = k3 - k + 3(k2 + k).   As 6 divides k3 - k and 3(k2 + k) is a multiple of 6, we conclude that (k+1)3 - (k+1) is also a multiple of 6.  By induction, 6 divides n3 - n for all integers n ≥ 0.  **Ex2.**Suppose you wish to prove that the following is true for all positive integers *n* by using the Principle of Mathematical Induction:  P(n) = “1 + 3 + 5 + …+ (2n – 1) = n2 ”  (a) Write *P*(1)  (b) Write *P*(12)  (c) Write *P*(13)  (d) Use the fact “*P*(12) is true” to prove “*P*(13) is true”  (e) Write *P*(*k*)  (f) Write *P*(*k* + 1)  (g) Use the Principle of Mathematical Induction to prove that *P*(*n*) is true for all positive integers *n.*  ***Solution.***  a/ “1 = 12”  b/ “1 + 3 + 5 + … + (2⋅12 – 1) = 122”  c/ “1 + 3 + 5 + … + (2⋅13 – 1) = 132”  d/ We have P(12) is true, or “1 + 3 + 5 + … + (2⋅12 – 1) = 122” is true.  So, 1 + 3 + 5 + … + (2⋅13 – 1)  = 1 + 3 + 5 + … + (2⋅12 – 1) + (2⋅13 – 1)  = 122 + (2.13 – 1) (due to the truth of P(12))  = 122 + (2.12 + 1)  = (12 + 1)2  = 132.  Hence, 1 + 3 + 5 + … + (2⋅13 – 1) = 132and P(13) is true.  e/ “1 + 3 + … + (2k – 1) = k2”  f/ “1 + 3 + … + [2(k + 1) – 1] = (k + 1)2”  g/   * BASIC STEP.   “1 = 12” 🡺P(1) is true.   * INDUCTIVE STEP.   Suppose for each positive integer k, P(k) is true, that is,  “1 + 3 + … + (2k – 1) = k2” is true.  Then, 1 + 3 + … + (2k – 1) + [2(k + 1) – 1] = k2 + [(2(k + 1) – 1)] (due to the truth of P(k))  = k2 + 2k + 1  = (k + 1)2  Hence, 1 + 3 + 5 + … + [2⋅(k+1) – 1] = (k + 1)2and P(k + 1) is true.  By induction, P(n) is true for all positive integers n.  **Ex3.**Use **strong induction** to prove that every amount of postage of six cents or more can be formed using 3-cent and 4-cent stamps.  ***Solution.***   * BASIS STEP. * 6 cents: two 3-cent stamps * 7 cents: one 3-cent stamp and one 4-cent stamp. * 8 cents: two 4-cent stamps. * INDUCTIVE STEP.   Assume every amount of postage of j cents (6 ≤ j ≤ k, k ≥ 8) can be formed using 3-cent and 4-cent stamps.  We need to show that an amount of postage of (k + 1) cents can be formed using 3-cent and 4-cent stamps.  In fact, k + 1 = (k – 2) + 3, and since 6 ≤ (k – 2) ≤ k, it follows that (k – 2) cents can be formed using 3-cent and 4-cent stamps (by the assumption above).  So, (k – 2) + 3 cents can be formed using 3-cent and 4-cent stamps. | **61/** Prove that 2 divides n2 + n whenever n is a positive integer.  **62/** Prove that 2n< n! if n is an integer greater than 3.  **63/**Suppose you wish to use the Principle of Mathematical Induction to prove that  1⋅1! + 2⋅2! + 3⋅3! + … + n⋅n! = (n+1)! – 1,  for all *n*≥ 1.  (a) Write *P*(1)  (b) Write *P*(5)  (c) Use *P*(5) to prove P(6)  (d) Write *P*(*k*)  (e) Write *P*(*k*+ 1)  (f) Use the Principle of Mathematical Induction to prove that *P*(*n*) is true for all *n*≥ 1.  **64/**Suppose that the only currency were 2-VND bills and 5-VND bills. Use **strong induction** to show that any amount greater than 3VND could be made from a combination of these bills. | |
| Recursive definitions | **Ex1.**Give a **recursive definition** of each of these functions.  a/ f(n) = n, n = 1, 2, 3, …  b/ f(n) = 3n + 5, n = 0, 1, 2, …  ***Solution.***  a/ f(n) = n, n = 1, 2, 3, …  BASIS STEP.  f(1) = 1  RECURSIVE STEP.  For n > 1, f(n) = n   * f(n – 1) = n – 1 * f(n) = f(n – 1) + 1   b/ f(n) = 3n + 5, n = 0, 1, 2, …  BASIS STEP.  f(0) = 5  RECURSIVE STEP.  For n > 0, f(n) = 3n + 5   * f(n – 1) = 3(n – 1) + 5 = 3n + 2   f(n) = f(n – 1) + 3  **Ex2.**Give a **recursive definition** of each of these sets.  a/ A = {2, 5, 8, 11, 14, …}.  b/ B = {…, -5, -1, 3, 7, 10, …}.  c/ C = {3, 12, 48, 192, 768, …}.  ***Solution.***  a/ A = {2, 5, 8, 11, 14, …}  BASIS STEP.  2 ∈ A  RECUSIVE STEP.  x∈ A → x + 3 ∈ A.  b/ B = {…, -5, -1, 3, 7, 10, …}  BASIS STEP.  3 ∈ B  RECUSIVE STEP.  x∈ B → (x + 4 ∈ B and x – 4 ∈ B).  c/ C = {3, 12, 48, 192, 768, …}  BASIS STEP.  3 ∈ C  RECUSIVE STEP.  x ∈ C → 4x ∈ C. | **65/**Give a **recursive definition** of each of these functions.  a/ f(n) = (-1)n, n = 0, 1, 2, 3, …  b/ f(n) = 7, for all n = 1, 2, 3, …  c/ f(n) = 1 + 2 + 3 + … + n, n =1, 2, 3, …  **66/**Find f(3), f(4) if:  a/ f(1) = 3 and f(n) = 2f(n-1) + 5.  b/ f(n) = f(n-1).f(n-2) and f(0) = 1, f(1) = 2.  c/ f(n) = (f(n-1))2 – 1 and f(1) = 2.  **67/**Give a **recursive definition** of each of these sets.  a/ A = {0, 3, 6, 9, 12, …}  b/ B = {…, -8, -4, 0, 4, 8, …}  c/ C = {0.9, 0.09, 0.009, 0.0009, …}. | |
| Recursive algorithms | **Ex1.**Consider an **recursive algorithm** to compute the nth Fibonacci number:  procedure Fibo(n : positive integer)  if n = 1 return 1  else if n = 2 return 1  else return Fibo(n – 1) + Fibo(n – 2)  How many additions (+) are used to find Fibo(6) by the algorithm above?  ***Solution.***  From the tree below, there are 7 additions.    **Ex2.**a/Give a **recursive algorithm** to find Sm(n) = m + n, where n is a non-negative integer and m is an integer.  b/ Use mathematical induction to show that the algorithm is correct.  ***Solution.*** a/   * Recursive definition of Sm(n):   BASIS STEP.  Sm(0) = m + 0 = m  RECURSIVE STEP.  For n > 0, Sm(n) = m + n   * Sm(n - 1) = m + (n – 1) * Sm(n) = Sm(n - 1) + 1 * Recursive algorithm to find Sm(n):   procedure sum(m: integer; n: non-negative integer)  if n = 0 then sum(m, n) := m  else then sum(m, n): =sum(m, n – 1) + 1  b/ Prove the correctness of the algorithm:  BASIS STEP.  If n = 0: sum(m, n) := m = m + 0 = m + n = Sm(n).  INDUCTIVE STEP.  Suppose for every integer k ≥ 0, sum(m, k) returns m + k.  We need to show that sum(m, k + 1) returns m + k + 1.  In fact, from the algorithm, k + 1 > 0 and sum(m, k + 1): = sum(m, k) + 1 and then returns m + k + 1.  (by the assumption, sum(m, k) returns m + k). | **68/**Consider an algorithm:  procedure Fibo(n : positive integer)  if n = 1 return 1  else if n = 2 return 1  else if n = 3 return 2  else return Fibo(n – 1) + Fibo(n – 2) + Fibo(n – 3)  How many additions (+) are used to find Fibo(6) by the algorithm above?  **69**a/Write a **recursive algorithm** to find the sum of first n positive integers.  b/ Use **mathematical induction** to prove that the algorithm in (a) is correct.  c/ Write a **recursive algorithm** to find the value of the function f(n) = 7, for n = 1, 2, 3, …  **70/**Consider the following algorithm:  procedure tinh(a: real number; n: positive integer)  if n = 1 return a  else return a⋅tinh(a, n-1).  a/ What is the output if inputs are: n = 4, a = 2.5? Explain your answer.  b/ Show that the algorithm computes n⋅a using Mathematical Induction. | |
| **Applications.**   1. Determine whether each of the following bit strings belongs to the set S **recursively defined** by:  * BASIS STEP: 0 ∈S * RECURSIVE STEP: 1w∈S or 0w ∈S if w∈S   a/ λ (the empty string)  b/ 0  c/ 110  d/ 10110   1. Let S be set of all bit strings of any length. Define the number #0(s) **recursively** by:  * *Basis step:* #0(s) = 0, where λ is the empty string. * *Recursive step:*   .   1. Find 2. Find 3. What can we say about s if ? 4. If s and w are two bit strings, show that . | | | |
| Chapter 5 – Counting | | | |
| Product rule & sum rule  Counting functions  Counting one-to-one functions | **Ex1.**Find the number of strings of length 7 of letters of the alphabet, with no repeated letters, that beginwith a vowel.  ***Solution.***   * Keep in mind a row of *seven blanks*:   - - - - - - -.   * There are *ﬁve ways* in which the ﬁrst letter in the string can be a vowel. * Once the vowel is placed in the ﬁrst blank, there are 25 ways in which to ﬁll in the second blank, 24 ways to ﬁll in the third blank, etc. * Using the product rule, we obtain     **Ex2.**Find the number of strings of length 7 of letters of the alphabet, with no repeated letters, that begin with C or V and end with C or V.  ***Solution.***  Using a row of 7 blanks, we ﬁrst count the number of strings belonging one of two cases:   * Case 1: Strings begin with C and end with V: C - - - - - V. * By the product rule, the number of ways to ﬁll in the five interior letters is 24 · 23 · 22 · 21 · 20. * Case 2: Strings begin with V and end with C: V - - - - - C. * By the product rule, the number of ways to ﬁll in the five interior letters is 24 · 23 · 22 · 21 · 20.   Therefore, by the **sum rule,** the answer is (24 · 23 · 22 · 21 · 20) + (24 · 23 · 22 · 21 · 20) = 2(24 · 23 · 22 · 21 · 20).  **Ex3.**How many subsets of the set {1, 2, 3, 4, 5}  a/ contain 2 and 3?  b/ do not contain 3?  c/ have more than one element?  ***Solution.***  a/ Suppose A is a subset of {1, 2, 3, 4, 5}, then A contains members chosen from {1, 2, 3, 4, 5}. We can see:   * 1 may belong to A or not. * 2 may belong to A or not. * 3 may belong to A or not. * 4 may belong to A or not. * 5 may belong to A or not.   Therefore, there 2⋅2⋅2⋅2⋅2 ways to construct A.   * There are 25 = 32 subsets of {1, 2, 3, 4, 5}.   b/ Similarly to the part a/, there are 24 = 16 subsets of {1, 2, 3, 4, 5} do not contain 3.  c/ number of subsets having more than one element = number of all subsets – number of subsets having no element = 25 – 1 = 31.  **Ex4.a/** How many functions are there from the set {a, b, c, d} to the set {1, 2, 3, 4}?  **a/** How many one-to-one functions are there from the set {a, b, c, d} to the set {1, 2, 3, 4}?  ***Solution.***  a/ Afunction corresponds to a choice of one of the 4 elements in the codomain{1, 2, 3, 4} for each ofelements {a, b, c, d}in the domain. Therefore, we have:   * 4 ways to choose the value of the function at a. * 4 ways to choose the value of the function at b. * 4 ways to choose the value of the function at c. * 4 ways to choose the value of the function at d. * By the product rule, there are 44 functions.   b/ An one-to-one function corresponds to a choice of one of the 4 elements in the codomain{1, 2, 3, 4} for each ofelements {a, b, c, d}in the domain so that no value of the codomain can be used again.  Therefore, we have:   * 4 ways to choose the value of the function at a. * 3 ways to choose the value of the function at b (because the value used for a cannot be used again for b). * 2 ways to choose the value of the function at c. * 1 ways to choose the value of the function at d. * Therefore, there are 4⋅3⋅2⋅1 one-to-one functions. | **71/**There are three available ﬂights from Hanoi to Bangkok and, regardless of which of these ﬂightsis taken,there are ﬁve available ﬂights from Bangkokto Manila. In how many ways can a person ﬂy from Hanoi to Manila via Bangkok?  **72/**Find the number of strings of length 7 of letters of the alphabet, with repeated letters allowed, thathave vowels in the ﬁrst two positions.  **73/**Find the number of strings of length 7 of letters of the alphabet, with no repeated letters, that begin with E and end with V or a vowel.  **74/**A final test of the course MAD101 contains 50 multiple choice questions. There arefour possible answers for each question.  a/ In how many ways can a student answer the questionsif the student answers every question?  b/ In how many ways can a student answer the questionson the test if the student can leave answers blank?  **75/** How many **subsets** of the set {(0, 0), (0, 1), (1, 0), (1, 1)}  a/ are there in total?  b/ contain (0, 0) and (1, 1)?  **76/a/** How many functions are there from the set {a, b, c, d} to the set {1, 2, 3}?  **b/** How many **one-to-one** functions are there from the set {a, b, c, d} to the set {1, 2, 3}? | |
| |A ∪ B| =  |A| + |B| - |A∩B|  (in A or in B) | **Ex1.**Find the number of integers from 100 to 1000 inclusive that are  a/ divisible by 7.  b/ divisible by 7 or 11.  ***Solution.***  a/ When we divide 1000by 7, we obtain142 + 6/7. Then, the largest integer in our range that is divisible by 7 is 142⋅7, or 994.  And if we divide 100 by 7, the result is about 14 + 2/7. So, the smallest integer in 100..1000that is divisible by 7 is 21, not 14.  Therefore, the number of integers between 100 and 1000 inclusive that are divisible by 7 is (994 – 21)/7 + 1, or 139.  b/   * From the part a/, there are 139 integers that are divisible by 7. * Similarly, there are (990 – 110)/11 + 1, or 81 integers between 100 and 1000 inclusive that are divisible by 11. * And again, there are (924 – 154)/77 + 1, or 11 integers between 100 and 1000 inclusive that are divisible by 77.   By Inclusion – Exclusion principle, the answer is 139 + 81 – 11 = 209.  **Ex2.**Find the number of strings of length 7 of letters of the alphabet, with no repeated letters, that begin with E or end with a vowel.  ***Solution.***  Using a row of seven blanks: - - - - - - -   * There are 25⋅24⋅23⋅22⋅21⋅20 strings of the form E- - - - - -. * There are 25⋅24⋅23⋅22⋅21⋅20⋅5 strings of the form - - - - - - (a vowel) * There are 24⋅23⋅22⋅21⋅20⋅4 strings of the form E- - - - - (a vowel, not E)   By Inclusion – Exclusion principle, the answer is 25⋅24⋅23⋅22⋅21⋅20 + 25⋅24⋅23⋅22⋅21⋅20⋅5 – 24⋅23⋅22⋅21⋅20⋅4. | **77/**Find the number of integers from 999 to 9999 inclusive that are:  a/ divisible by 13 or 17.  b/ divisible by 13but not by 17.  **78/**Find the number of strings of length 7 of letters of the alphabet, with no repeated letters, that  a/ begin with V or end with a vowel.  b/ begin or end with a vowel.  c/ begin or end with a vowel (but not both). | |
| Counting problems and Recurrence relations | **Ex1.**A vending machine dispensing books of stamps acceptsonly one-dollar coins, $1 bills, and $5 bills.  a/ Find a recurrence relation for the number of ways to deposit n dollars in the vending machine, where the order in which the coins and bills are deposited matters.  b/ Find a0, a1, a2, a3, a4.  c/ How many ways are there to deposit $7 for a bookof stamps?  ***Solution.***  a/ Let an be the number of ways to deposit n dollars in the vending machine.  Some ways to deposit n dollars:   * One $1coin first, then (n – 1) dollars. In this case, there are an-1 ways corresponding to (n – 1) remaining dollars. * One $1 bill first, then (n – 1) dollars. In this case, there are also an-1 ways corresponding to (n – 1) remaining dollars. * One $5 bill first, then (n – 5) dollars (if n > 5). In this case, there are an-5 ways corresponding to (n – 5) remaining dollars.   So, we have the recurrence relation  an = 2an-1, if 5 > n ≥ 1  an = 2an-1 + an-5, if n ≥ 5  b/   * a0 = 1 // the only way to deposit zero dollar is depositing nothing. * a1 = 2a0 = 2. * a2 = 2a1 = 4   $1-coin, $1-bill  $1-bill, $1-coin  $1-coin, $1-coin  $1-bill, $1-bill   * a3 = 2a2 = 2.4 = 8 * a4 = 2a3 = 16.   c/ a5 = 2a4 + a0 = 32 + 1 = 33  a6 = 2a5 + a1 = 66 + 2 = 68  a7 = 2a6 + a2 = 136 + 4 = 140.  **Ex2.**Find a **recurrence relation** for the number of bit stringsof length n that do not contain three consecutive 0s.  ***Solution.***  Let an be the number of bit stringsof length n that do not contain three consecutive 0s.  For example, a1 = 2 (two bit strings “0” and “1” of length 1), a2 = 4 and a3 = 7 (except for the string “000”).  Strings of length n we want to count are of exactly one of three cases:   * 1 (n – 1 remaining bits satisfying the condition). For example, with n = length = 4, 1001 and 1100are strings of this type, but 1~~000~~ or ~~0~~110 are not. 🡺there are an-1 such strings. * 01 (n – 2 remaining bits satisfying the condition) 🡺 there are an-2 such strings. * 001(n – 3 remaining bits satisfying the condition)🡺 there are an-3 such strings.   Therefore, an = an-1 + an-2 + an-3.  **Ex3.**Verify that *an*= 3*n*+2 is a solution to the recurrence relation *an*= 4*an*- 1- 3*an*- 2.  ***Solution.***  an = 3n + 2   * an-1 = 3(n-1)+2 = 3n+1 * an-2 = 3(n-2)+2 = 3n * 4*an*- 1- 3*an*- 2 = 4.(3n+1) – 3(3n) * 4*an*- 1- 3*an*- 2 = 3.(3n+1) + (3n+1) – 3(3n) = 3.(3n+1) = 3n+2 * 4*an*- 1- 3*an*- 2 = an.   Therefore, *an*= 3*n*+2 is a solution to the recurrence relation *an*= 4*an*- 1- 3*an*- 2.  **Ex4.**You take a job that pays $25,000 annually.How much do you earn *n* years from now if you receive a four percent raise each year?  ***Solution.***  a/ Let Sn be the salary after n years.  Then, Sn = (1 + 0.04)Sn-1   * Sn = (1 + 0.04)nS0 = 25,000⋅(1.04). | **79/**How many bit strings of length eightdo not contain three consecutive 0s?  **80/**Verify that *an* = 3*n*and *an* = 3*n*+1 are solutions to the recurrence relation *an* = 4*an* - 1- 3*an* - 2.  **81/**You take a job that pays $10,000 annually.  a/ How much do you earn *20* years from now if you receive a ten percent raise each year?  b/ How much do you earn *20* years from now if each year you receive a raise of $1000 plus four percent of your previous year's salary? | |
| **Applications.**  **1/**Messages are transmitted over a communications channel using two signals. The transmittal of one signal requires1 microsecond, and the transmittal of the other signal requires 2 microseconds.  a/ Find a **recurrence relation** for the number of different messages consisting of sequences of these two signals, where each signal in the message is immediately followed by the next signal, that can be sent in n microseconds.  b/ What are the initial conditions?  c/ How many different messages can be sent in 10 microseconds using these two signals?  **2/**Suppose inflation continues at five percent annually. (That is, an item that costs $1.00 now will cost $1.05 next year). Let *an*= the value (that is, the purchasing power) of one dollar after *n* years.  a/ Find a **recurrence relation** for *an*.  b/ What is the value of $1000 after 10 years?  c/ What is the value of $1000 after 50 years?  d/ If inflation were to continue at ten percent annually, find the value of $1000 after 50 years. | | | |
| Chapter 8 - Relations | | | |
| Binary relation  Properties of relations  Combination of relations  Composite relation | **Ex1.**List the ordered pairs in the **relation** R fromA ={0, 1, 2, 3, 4} to B ={0, 1, 2, 3}, where (a, b) ∈ Rif and only if  a/ a + b = 4.  b/ a | b.  ***Solution.***  a/ R = {(1, 3); (2, 2); (3, 1); (4, 0)}  b/ R = {(1, 1); (1, 2); (1, 3); (1, 0); (2, 0); (2, 2); (3, 0); (3, 3)}.  **Ex2.**Determine whether the relation R on the set of all realnumbers is **reﬂexive, symmetric, antisymmetric, and/or transitive**, where (x, y)∈R if and only if  a/ (x, y)∈R ⇔x = 2y.  b/ x = 1.  ***Solution.***  a/ x = 2y.   * (1, 1)∉R (because 1 ≠ 2.1) 🡺 R is not reflexive. * 2 = 2.1 🡺 (2, 1)∈R but (1, 2)∉R (because 1 ≠ 2.2) 🡺 R is not symmetric. * If xRy and yRx🡺 x = 2y and y = 2x 🡺 x = y (= 0) 🡺 R is antisymmetric. * (4, 2)∈R and (2, 1)∈Rbut (4, 1)∉R 🡺 R is not transitive.   b/ (x, y)∈R ⇔ x = 1.   * (2, 2) ∉R 🡺 R is not reflexive. * (1, 2) ∈R but (2, 1) ∉R 🡺 R is not symmetric. * If (x, y)∈R and (y, x)∈R, then x = 1 and y = 1 🡺 x = y.   Hence, R is antisymmetric.   * If (x, y)∈R and (y, z)∈R, then x = 1 and y = 1 🡺 (x, z)∈R.   Hence, R is transitive.  **Ex3.**Let R be the relation on the set of ordered pairs of positive integers such that (a, b)R(c, d)⇔a + d = b + c.  Show that  a/ R is reflexive.  b/ R is symmetric.  c/ R is transitive.  ***Solution.***  a/ For every positive integer a, (a, a) R (a, a) because a + a = a + a.  b/ (a, b)R(c, d)⇔a + d = b + c  ⇔ c + b = d + a ⇔ (c, d)R(a, b).  Hence, R is symmetric.  c/ For all positive integers a, b, c, d, m and n, if (a, b)R(c, d) and (c, d)R(m, n), then  a + d = b + c and c + n = d + m   * a + d + c + n = b + c + d + m * a + n = b + m * (a, b)R(m, n).   Therefore, R is transitive.  **Ex4.**Let R = {(1, 1), (3, 3), (2, 3)},and S = {(1, 2), (3, 1), (2, 2)} be relations on the set {1, 2, 3}. Find  a/ R – S.  b/ R ∩ S.  c/ R ∪ S.  d/ R ⊕ S.  e/  f/ S-1.  g/ SR.  ***Solution.***  V | | **82/**List the ordered pairs in the **relation** R fromA ={0, 1, 2, 3, 4} to B ={0, 1, 2, 3}, where (a, b) ∈ Rif and only if  a/ a > b.  b/ a – b = 1.  c/ a = 2b.  **83/**Determine whether the relation R on the set of all realnumbers is **reﬂexive, symmetric, antisymmetric, and/or transitive**, where (x, y)∈Rif and only if  a/ xy = 0.  b/ x = y + 1 or x = y – 1.  c/ x ≡ y (mod 5).  **84/**Let R be the relation on the set of ordered pairs of positive integers such that  (a, b)R(c, d) ⇔ad = bc.  Show that  a/ R is reflexive.  b/ R is symmetric.  c/ R is transitive.  **85/**Let R = {(1, 2), (1, 3), (2, 3), (3, 1)},and S = {(2, 1), (3, 1), (3, 2)} be relations on the set {1, 2, 3}. Find  a/ R – S.  b/ R ∩ S.  c/ R ∪ S.  d/ R ⊕ S.  e/  f/ S-1.  g/ SR.  **86/**List the 16 different relations on the set {0, 1}.  **87/**Which of the 16 relations on {0, 1}, are  a/ reﬂexive?  b/ ir-reﬂexive?  c/ symmetric?  d/ anti-symmetric?  e/ asymmetric?  f/ transitive? |
| Counting relations | **Ex1.**How many different relations on {a, b} containthe pair (a, b)?  ***Solution.***  Every relation on the set {a, b} is a subset of the Cartesian product {a, b}× {a, b}.  On other hand, {a, b}× {a, b} = {(a, a); (a, b); (b, a); (b, b)}, which has 24 subsets.   * There are 24 = 16 relations.   **Ex2.**How many different **reflexive relations** are there on the set {a, b}?  ***Solution.***  Every relation on the set {a, b} is a subset of the Cartesian product {a, b}× {a, b}.  And a **reflexive relations**on the set {a, b} is a set containing both (a, a); and (b, b). By the product rule, there are 1.1.2.2 such subsets.  Therefore, there are 4 **reflexive** relations on the set {a, b}.  **Ex3.**How many different relations are there from {a, b, c, d} to {1, 2, 3}?  ***Solution.***  There are 24⋅3 = 212 relations from {a, b, c, d} to {1, 2, 3}. | | **88/**a/ How many different relations are there on the set {a, b, c}?  b/ How many different relations on the set {a, b, c} do not contain (a, a)?  c/ How many different **ir-reflexive relations** are there on the set {a, b, c}? |
| Representations of relations | **Ex1.**Represent each of these relations on {1, 2, 3, 4} with a matrix (with the elements of this set listed in increasing order).  a/ {(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)}.  b/ {(1, 1), (1, 4), (2, 2), (3, 3), (4, 1)}.  ***Solution.***  a/  .  b/ .  **Ex2.**How many 1-entries does the matrix representing the relation R on A ={1, 2, 3,..., 100} consisting ofthe ﬁrst 100 positive integers have if R is  a/ {(a, b) | a ≤ b}?  b/ {(a, b) | a + b = 100}?  ***Solution.***  a/aRb⇔ a ≤ b.     * The number of 1-entries is (1 + 2 + 3 + … + 100) = 5050.   b/ aRb⇔ a + b = 100.  The matrix has the size of 100x100.  Since the (row i, column 100 – i)-position in the matrix is the only 1-entry of row i, it follows that the matrix has n 1-entries.  **Ex3.**Let R be the relation represented by the matrix  .  Find the matrix representing  a/ R-1.  b/ .  c/ R2.  d/ R – R2.  e/ R⊕R2.  ***Solution.***  a/ Let MR and be the matrices representing relations R and R-1.  Recall that (i, j) ∈R-1⇔ (j, i)∈R, or equivalently,  (i, j)-entry = 1 in ⇔(j, i)-entry = 1 in MR.   * is the transpose of MR.   b/ Let MR and be the matrices representing relations R and.  Recall that (i, j)∈⇔(i, j) ∉R, or equivalently,  (i, j)-entry = 1 in ⇔ (i, j)-entry = 0 in MR.   * .   c/ Let be the matrix representing the relation R2.  The matrix of R2 (= RR) can be computed by Boolean product of MRMR    d/ Let be the matrix representing relation R – R2.  Recall that A – B is the set of elements that belong to A but not belong to B.  So, the relation R – R2 contains only ordered pairs (a, b) where (a, b)∈R but (a, b)∉R2.  So, the (i, j)-entry of is 1 ⇔ the (i, j)-entry of MR is 1 and the (i, j)-entry of  is 0.  Therefore,  e/ Let be the matrix representing the relation R⊕R2.  Recall that R⊕R2 contains only ordered pairs (a, b) that belong to exactly one of (R – R2) and (R2 – R).  So, (i, j)-entry of is 1 ⇔ (i, j)-entry of is 1 OR (i, j)-entry ofis 1 (BUT NOT BOTH).  Therefore,  **Ex4.**Suppose that the relation R on a set is represented by the matrix.  a/ Is R reflexive?  b/ Is R symmetric?  c/ Is R antisymmetric?  ***Solution.***  a/ Recall that a relation R on a set A is reflexive if and only if  ∀a∈A, (a, a)∈R.  Or equivalently, in the matrix MR, the  (row i, column i)-entry is 1 for every value of i.  We can see (3, 3)-entry of MR is 0 🡺 (3, 3) ∉R 🡺 R is not reflexive.  b/ Recall that a relation R on a set A is symmetric if and only if  ∀a∀b, (a, b)∈R → (b, a) ∈R.  Based on this definition, R is symmetric if and only if the matrix MR is symmetric, that is, the (i, j)-entry of MR equals to the (j, i)-entry of MR.  Since MR is not symmetric ((1, 2)-entry of MR is 1 and (2, 1)-entry of MR is 0), we can conclude that R is not symmetric.  c/ Recall that the relation R is antisymmetric if and only if (a, b) ∈ R and (b, a) ∈ R imply that a = b.Consequently, the matrix of anantisymmetric relation has the property that if mij = 1 with i ≠ j, then mji = 0. Or, in other words, either mij = 0or mji = 0 when i ≠ j .  So, it is easy to see that R is not antisymmetric (m23 = m32 = 1). | | **89/**Represent each of these relations on {1, 2, 3, 4} with amatrix (with the elements of this set listed in increasingorder).  a/ {(1, 1), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4), (3, 1), (4, 1), (4, 2)}  b/ {(1, 4), (3, 1), (3, 2), (3, 4)}.  **90/**How many 1- entries does the matrix representingthe relation R on A ={1, 2, 3,..., 100} consisting ofthe ﬁrst 100 positive integers have if R is  a/ {(a, b) | a = b ± 1}?  b/ {(a, b) | a + b < 101}?  **91/** Let R and S be relations on a set represented by the matrices  and.  Find the matrices that represent  a/ R∪S.  b/ R∩S-1.  c/ R – .  d/ R⊕S.  e/ RS.  **92/**Suppose that the relation R on a set is represented by the matrix.  a/ Is R reflexive?  b/ Is R symmetric?  c/ Is R antisymmetric? |
| Equivalence relations  Partitions & equivalence classes | **Ex1.**Let R be the relation on the set of real numbers such that  aRb *if and only if a − b is an integer*.  Show that R is an **equivalence relation**.  ***Solution.***   * Because a −a = 0 is an integer for all real numbers a, aRa for all real numbers a. Hence, R is reﬂexive. * Now suppose that aRb. Then a −b is an integer, so b−a is also an integer. Hence, bRa. It follows that R is symmetric. * If aRb and bRc, then a −b and b−c are integers. Therefore, a −c = (a −b) + (b−c) is also an integer. Hence, aRc. Thus, R is transitive.   Consequently, R is an **equivalence relation**.  **Ex2.(Congruence Modulo m).** Let m be an integer with m> 1. Show that the relation  R ={(a, b) | a ≡ b(mod m)}is an **equivalence relation** on the set of integers.  ***Solution.***  Recall that a ≡ b (mod m) if and only if m divides a − b.   * Note that a − a = 0 is divisible by m, because 0 = 0 · m. Hence, a ≡ a (mod m), so congruence modulo m is reﬂexive. * Now suppose that a ≡ b (mod m). Then a − b is divisible by m, so a − b = km, where k is an integer. It follows that b − a = (−k)m,so b ≡ a (mod m). Hence, congruence modulo m is symmetric. * Next, suppose that a ≡ b (mod m) and b ≡ c (mod m). Then m divides both a − b and b − c. Therefore, there are integers k and l with a − b = km and b − c = lm. Adding these two equations shows that a − c = (a − b) + (b − c) = km + lm = (k + l)m. Thus, a ≡ c (mod m). Therefore, congruence modulo m is transitive.   It follows thatcongruence modulo m is an **equivalence relation**.  **Ex3.**List the ordered pairs in the **equivalence relation** R produced by the **partition** A1={1, 2},A2 ={3}, and A3 ={4} of S ={1, 2, 3, 4}.  ***Solution.***  R = {(1, 1); (1, 2); (2, 1); (2, 2); (3, 3); (4, 4)}.  **Ex4.**Determine whether the relation with thedirected graph shown is an **equivalence relation** on the set {a, b, c, d}.    ***Solution.***   * R is reflexive (there are loops at every vertex). * R is symmetric (there is an edge from v1 to v2 whenever there is an edge from v2 to v1). * R is transitive (if there is an edge from v1 to v2 and an edge from v2 to v3, then there is an edge from v1 to v3).   So, R is an equivalence relation.  **Ex5.**Which of these relations on {0, 1, 2, 3} are **equivalence relations**? What are the **equivalence classes** of that equivalence relation?  a/R = {(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)}.  b/S = {(0, 0), (1, 2), (2, 1), (2, 2), (2, 3), (3, 3)}.  ***Solution.***  a/ R is reflexive, symmetric and transitive. So, R is an equivalence relation.  Equivalence classes are: {0}; {1, 2}; {3}.  b/ S is not an equivalence relation because S is not symmetric ((2, 3)∈S but (3, 2)∉S). Therefore, S is not an equivalence relation. | | **93/**Which of these relations on {0, 1, 2, 3} are **equivalence relations**? What are the **equivalence classes** of that equivalence relation?  a/R = {(0, 0), (1, 1), (2, 2), (3, 3)}.  b/ S = {(0, 0), (1, 1), (1, 2), (2, 1), (2, 2), (3, 3)}.  c/T = {(0, 0), (1, 1), (1, 3), (2, 2), (2, 3), (3, 1), (3, 2),  (3, 3)}.  **94/**Suppose that A is a nonempty set, and f is a function thathas A as its domain. Let R be the relation on A consistingof all ordered pairs (x, y) such that f(x) = f(y).  Show that R is an **equivalence relation** on A.  **95/**Which of these relations on the set of all people are **equivalence relations**? Determine the properties of an equivalence relation that the others lack.  a/ {(a, b) | a and b are the same age}.  b/ {(a, b) | a and b share a common parent}.  c/ {(a, b) | a and b speak a common language}.  **96/**What is the **congruence class [3]m** (that is, the **equivalence class** of 4 with respect to congruence modulo m) when m is  a/ 2  b/ 3  c/ 4  d/ 5  **97/**List the ordered pairs in the **equivalence relations** produced by the**partition**{a,b}, {c, d}, {e} of {a, b, c, d, e}.  **98/**Determine whether the relation with thedirected graph shown is an **equivalence relation**. |
| n-ary relations and application to database | **Ex1.**The 3-tuples in a 3-ary relation represent the followingattributes of a student database: student IDnumber, name,phone number.What is a likely **primary key** for this relation?  ***Solution.***   * Two students may have the same name 🡺is not a likely **primary key** for this relation. * Some students may not have phone numbers 🡺 phone number is also not a likely **primary key** for this relation. * StudentshavedifferentIDnumbers🡺 student ID number is a likely **primary key** for this relation.   **Ex2.**What do you obtain when you apply the **projection** P2,3,5to the 5-tuple (a,b,c,d,e)?  ***Solution.***  P2,3,5(a,b,c,d,e) = (a, d). | | **99/**The 4-tuples in a 4-ary relation represent these attributesof published books: title, ISBN, publication date, numberof pages.What is a likely **primary key** for this relation?  **100/**Which **projection** mapping is used to delete the ﬁrst,second, and fourth components of a 6-tuple? |
|  | Applications. | |  |
|  | Chapter 9 – Graph Theory | |  |
| Simple graphs  Edge  Vertex/vertices  Special simple graphs: Kn, Cn, Wn, Qn  Handshaking theorem.  Degree of a vertex  Adjacent  Incident | **Ex1.**The **degree sequence** of a graph is the sequence of the degrees of the vertices of the graph in non-increasing order. Howmany **edges** does a graph have if its **degree** sequenceis 4, 3, 3, 2, 2?  ***Solution.***  Based on the **handshaking theorem**, number of edges = (½)(the sum of degrees of vertices) = (½)(4 + 3 + 3 + 2 + 2) = 7.  **Ex2.**A sequence d1,d2,...,dn is called **graphic** if it is the **degree sequence** of a **simple graph**. Determine whether each of these sequences is **graphic**.For those that are, draw a graph having the given degreesequence.  a/ 3, 3, 3, 3, 2, 0.  b/ 5, 4, 3, 2, 1.  c/ 7, 6, 5, 4, 4, 2, 1, 1.  ***Solution.***  Recall that a simple graph has no any a multiple edge or a loop.  a/ Below is a simple graph having the degree sequence 3, 3, 3, 3, 2, 0.    b/ Recall that a graph cannot have an odd number of vertices that have odd degrees.So, no graph having the degree sequence **5**, 4, **3**, 2, **1** (3 vertices that have odd degrees).  c/ Suppose there is such a **simple graph** with vertices a, b, c, d, e, f, g, h where deg(a) = 7, deg(b) =6, deg(c) = 5, deg(d) = 4, deg(e) = 4, deg(f) = 3, deg(g) = 1 and deg(h) = 1.   * First, vertex a must be adjacent to 7 other vertices. Hence, vertex a is adjacent to both g and h. * Next, there are 6 vertices that are adjacent to b. From 7 remaining vertices beside b, at least one of g and h is adjacent to b. In this situation, at least one of g and h must have degree 2 or larger. It is a contradiction with the fact deg(g) = deg(h) = 1.   So, there is no such a simple graph.  **Ex4.**The **complementary graph**of a simple graph G hasthe same vertices as G. Two vertices are adjacent in G ifand only if they are not adjacent in G.  Draw the **complementary graph** of the graph below.    ***Solution.***  By the definition, the **complementary graph** is given below:    Note that the union of G and is the complete graph Kn, where n is the number of vertices of G. So, if G has m edges, then  has n(n-1)/2 – m edges. | | **101/**Draw these **simple graphs**.  a/ K7.  b/ C7.  c/ W7.  d/ K3,4.  **102/**Find the **degree sequence** of each of the graphs in the exercise 101.  **103/** An undirected graphhasfive **vertices**of **degree** three and three **vertices** of **degree** five. How many **edges** does the graph have?  **104/** Determine whether each of these sequences is**graphic**.For those that are, draw a graph having the given degreesequence.  a/ 5, 4, 3, 2, 1, 0.  b/ 1, 1, 1, 1.  c/ 4, 4, 3, 2, 1.  b/ 8, 8, 4, 4, 2, 2, 0, 0.  **105/**The **complementary graph**of a simple graph G hasthe same vertices as G. Two vertices are adjacent in G ifand only if they are not adjacent in G.  a/ If G is a **simple graph** with 9**vertices** and has 11**edges**,how many **edges** does G have?  b/ If the **degree sequence** of the **simple graph** G is  4, 2, 2, 1, 1, what is the **degree sequence** of G? Draw the graphs G and |
| Bipartite graphs | **Ex1.**For which values of n is thegraph Cn**bipartite**?  a/ Cn.  b/ Wn.  ***Solution.***  a/Cn.   * If n is even, “odd-position” vertices and “even-position” vertices must be colored by different color (e.g., red for “odd-position” and black for “even-position” vertices). Since no edge connects an “odd-position” vertex to an “even-position” vertex, the graph Cn is bipartite if n is even. * If n is odd, the first and the final vertices in the cycle are both in “odd-positions”and they connect each other. This means, they are colored by the same color while they are adjacent. Therefore, Cn is not bipartite if n is odd.   b/ Wn.  Since the vertex at the center connecting to all other vertices around the cycle, the graph Wn is a non-bipartite graph.  **Ex2.**Suppose that there are four employees in the computersupport group of the School of Engineering of a largeuniversity. Each employee will be assigned to supportone of four different areas: hardware, software, networking, andwireless. Suppose that Ping is qualiﬁed to supporthardware, networking, andwireless;Quiggley is qualiﬁedto support software and networking; Ruiz is qualiﬁed tosupport networking and wireless, and Sitea is qualiﬁed tosupport hardware and software.  Use a graph tomodel the four employees andtheir qualiﬁcations. Is this graph bipartite?  ***Solution.***  V  **Ex3.**  ***Solution.***  V | | For which values of n are these graphs **bipartite**?  a/ Kn.  b/ Qn.  Suppose that there are ﬁve young women and six youngmen on an island. Each woman is willing to marry someof the men on the island and each man is willing to marryany woman who is willing to marry him. Suppose that  Anna is willing to marry Jason, Larry, and Matt; Barbarais willing to marry Kevin and Larry; Carol is willing tomarry Jason, Nick, and Oscar; Diane is willing to marryJason, Larry, Nick, and Oscar; and Elizabeth is willing to marry Jason and Matt. Model the possible marriages on the island using agraph. Is this graph bipartite? |
| Circuits  Connected graphs  Cut vertex  Cut edge | **Ex1.**  ***Solution.***  V  **Ex2.**  ***Solution.***  V  **Ex3.**  ***Solution.***  V | |  |
| Representing graphs  Adjacency matrix  Incidence matrix | **Ex1.**Find an **adjacency matrix** for each of these graphs.  a/ K5.  b/ W5.  c/ K2,3.  ***Solution.***  a/  b/  c/  **Ex2.**Find the number of 1-entries in the **incidence matrix** of each of these graphs.  a/ K7.  b/ K2,5.  ***Solution.***  a/ Recall that the incidence matrix has the number of columns equaling to the number of edges of the graph.  So, the **incidence matrix** of the graph K7 has 7(8)/2 = 28 columns.  By the definition of an incidence matrix and because K7 is a simple graph, each column of this incidence matrix has exactly two 1-entries. Hence, there are 28⋅2 = 56 1-entries in this matrix.  b/ Similarly to part a/, since the graph K2,5 has 2⋅5 = 10 edges, the incidence matrix of this graph has 20 1-entries.  **Ex3.**  ***Solution.***  V | | Find an **adjacency matrix** for each of these graphs.  a/ K6.  b/ C6.  c/ W6.  d/ K2,4.  e/ Q3.  Find the number of 1-entries in the **incidence matrix** of each of these graphs.  a/ Kn.  b/ Wn.  c/ Km,n. |
| Isomorphism  Path of length n  Counting paths | **Ex1.**Use paths either to show that these graphs are not isomorphic or to ﬁnd an isomorphism between them.    ***Solution.***  We will show that two graphs are not isomorphic by using a special path the one graph has but another graph has not.  In fact, the left-hand side graph (G) has one path making a “triangle” (e.g. u1-u2-u3) while the graph H has no the same property.  So, two graphs are not isomorphic.  **Ex2.**How many paths of length 3 between A and B does the graph have?    ***Solution.***  The adjacency matrix of the graph is    To find the number of paths of length 3 between A and B, we can multiply the (1, 2)-entry of the matrix M3.  First, we will compute M2.    Next, to find the (1, 2)-entry of M3, we multiply the first row of M2 by the second column of M. The result is 4. | | Use paths either to show that these graphs are not isomorphic or to ﬁnd an isomorphism between them.    How many paths of length 2 between a andb does each of these graph have?  a/    b/ |
| Euler circuits  Hamilton circuits | **Ex1.**  ***Solution.***  V  **Ex2.**  ***Solution.***  V  **Ex3.**  ***Solution.***  V | |  |
| Shortest paths  Dijkstra’s algorithm | **Ex1.**  ***Solution.***  V  **Ex2.**  ***Solution.***  V  **Ex3.**  ***Solution.***  V | |  |
|  | Applications. | |  |
|  | Chapter 10 – trees | |  |
| Definition of tree | **Ex1.**  ***Solution.***  V  **Ex2.**  ***Solution.***  V  **Ex3.**  ***Solution.***  V | |  |
| Leaf  Internal nodes  Child  parent | **Ex1.**  ***Solution.***  V  **Ex2.**  ***Solution.***  V  **Ex3.**  ***Solution.***  V | |  |
| m-ary trees  full m-ary trees | **Ex1.**  ***Solution.***  V  **Ex2.**  ***Solution.***  V  **Ex3.**  ***Solution.***  V | |  |
| Properties | **Ex1.**  ***Solution.***  V  **Ex2.**  ***Solution.***  V  **Ex3.**  ***Solution.***  V | |  |
| Binary search trees | **Ex1.**  ***Solution.***  V  **Ex2.**  ***Solution.***  V  **Ex3.**  ***Solution.***  V | |  |
| Prefix codes | **Ex1.**  ***Solution.***  V  **Ex2.**  ***Solution.***  V  **Ex3.**  ***Solution.***  V | |  |
| Decision trees | **Ex1.**  ***Solution.***  V  **Ex2.**  ***Solution.***  V  **Ex3.**  ***Solution.***  V | |  |
| Huffman coding algorithm | **Ex1.**  ***Solution.***  V  **Ex2.**  ***Solution.***  V  **Ex3.**  ***Solution.***  V | |  |
| Tree traversal  Pre-order  In-order  Post-order | **Ex1.**  ***Solution.***  V  **Ex2.**  ***Solution.***  V  **Ex3.**  ***Solution.***  V | |  |
| Expression tree  Prefix  Infix  Postfix  notations | **Ex1.**  ***Solution.***  V  **Ex2.**  ***Solution.***  V  **Ex3.**  ***Solution.***  V | |  |
|  | Applications. | |  |
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